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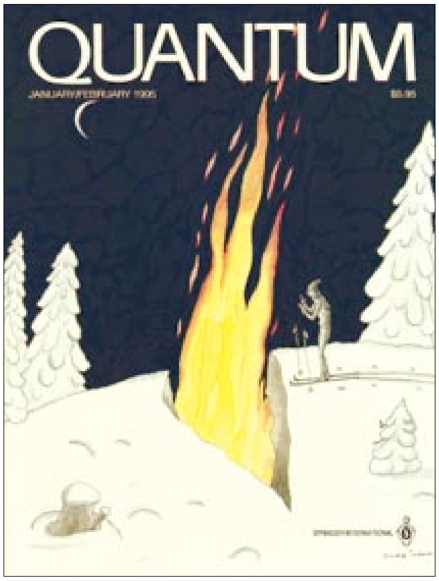


Cover art by Leonid Tishkov

While cross-country skiers may dream of a roaring fire in the middle of their trek, our planet generally keeps its inner heat to itself. Hot springs, geysers, and lava flows are relatively rare on the Earth's surface. It's not hard to figure out why the hot stuff stays bottled up in the planet's interior, or how it occasionally leaks out through cracks in the Earth's crust.

You may have a harder time explaining where the heat comes from. In "Taking the Earth's Temperature," Alexey Byalko explores this question, and along the way he uncovers some interesting facts about the Earth's thermal history and its present structure. In a companion piece, A. G. W. Cameron presents a theory for the creation of the Earth's little sister—the Moon.

We hope our readers in the northern climes have ample opportunity to engage in winter sports and outdoor activities in the coming months, whether or not they enjoy the amenity depicted on our cover. And we wish all our readers everywhere a healthy and happy New Year!

FEATURES

DEPARTMENTS

What is elegance?

Mathematicians say: "I know it when I see it"

by Julia Angwin

IN ARCHITECTURE, THE epitome of elegance might be a Greek temple. In fashion, a Chanel suit. In mathematics, it's a term applied to the best, shortest, most inspired (and inspirational) proofs.

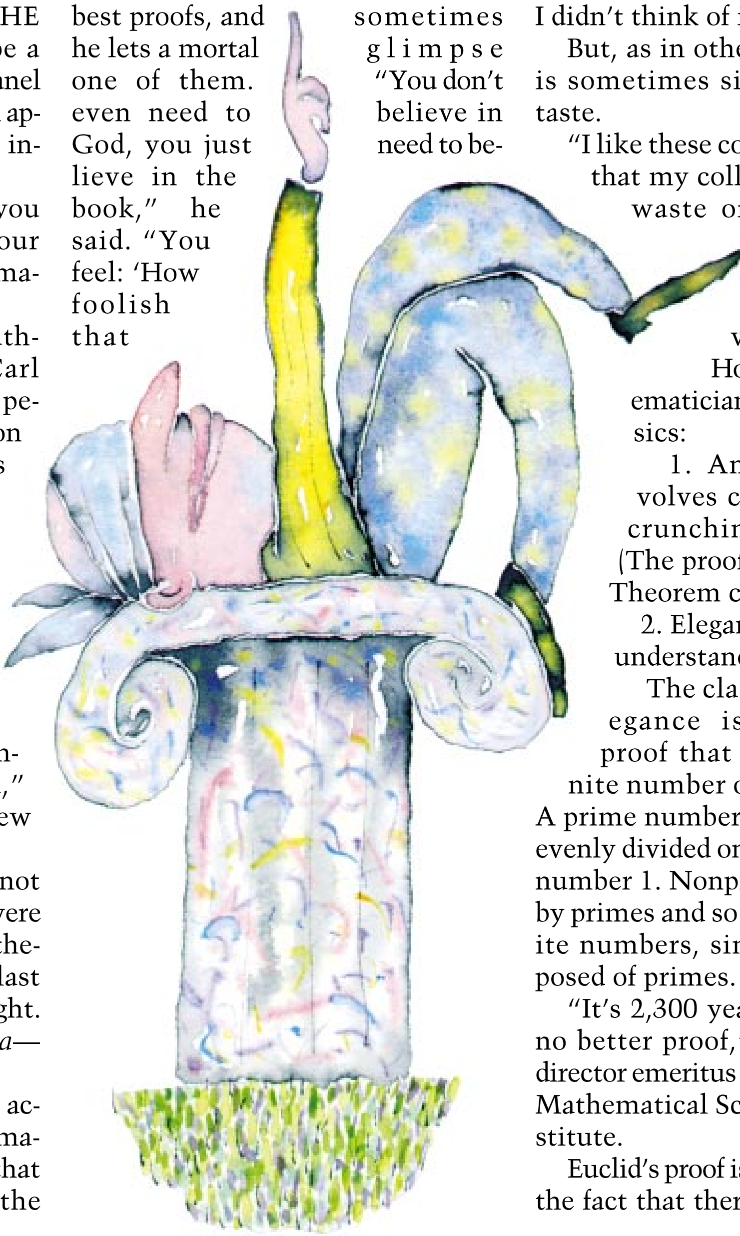
"An elegant proof just hits you between your eyes and fills your heart with joy," explains mathematician Irving Kaplansky.

One of the most elegant mathematicians of all time was Carl Friedrich Gauss. He lived in the period following the rapid expansion and development of mathematics in the 18th century. However, that century was not a period of elegance, according to mathematician Harold Edwards, who studies the history of math. It was Gauss in the 19th century who collected and refined the work done previously.

"He didn't publish anything until it was completely polished," says Edwards, a professor at New York University.

Gauss frustrated his peers by not publishing his proofs until they were perfect, but he thought that a cathedral is not a cathedral until the last scaffolding is down and out of sight. His motto was *Pauca sed matura*—"Few, but ripe."

Elegant proofs come from God, according to the Hungarian mathematician Paul Erdős. His theory is that God has a book containing all the best proofs, and he sometimes lets a mortal glimpse one of them. "You don't even need to believe in God, you just need to believe in the book," he said. "You feel: 'How foolish that I didn't think of it myself.'"



Art by Sergey Ivanov

But, as in other things, elegance is sometimes simply a matter of taste.

"I like these combinatorial things that my colleagues think are a waste of time," laments John Conway, professor at Princeton University.

However, most mathematicians agree on the basics:

1. Any proof that involves computer number-crunching is not elegant. (The proof of the Four Color Theorem comes to mind.)

2. Elegant proofs are easily understandable.

The classic example of elegance is Euclid's short proof that there are an infinite number of prime numbers. A prime number is one that can be evenly divided only by itself and the number 1. Nonprimes are divisible by primes and so are called composite numbers, since they are composed of primes.

"It's 2,300 years old and there's no better proof," says Kaplansky, director emeritus of the UC–Berkeley Mathematical Science Research Institute.

Euclid's proof is short and bypasses the fact that there is no formula for determining prime numbers. Without his proof, one could flounder around trying larger and larger primes without ever determining an upper limit. Euclid simply assumed that there is a final prime number and proves that this assumption leads to a contradiction. Here's the proof.

1. Call the highest prime number Q.

2. Now multiply Q by all the primes leading up to it: $2 \times 3 \times 5 \times 7 \times \ldots \times Q$.

3. Let P equal that product plus 1: $(2 \times 3 \times 5 \times 7 \times \ldots \times Q) + 1 = P$.

4. Then P is not divisible by any of the numbers 2, 3, 5, 7, …, Q, because each divisor would leave a remainder of 1.

5. But P must be divisible by some prime because it is a composite number.

6. But that prime must be larger than Q, because we have used up all the smaller primes. This contradicts step 1.

7. So the assumption must have been false—there must be an infinite number of primes.

Problem 1. Note that Euclid does not claim that his number P is a prime. Indeed, show that $2 \times 3 \times 5 \times 7 \times 11 \times 13 + 1$ is divisible by 59. Can you find a prime divisor of $2 \times 3 \times 5 \times 7 \times 11 \times 13 \times 17 + 1$?

Part of the charm of Euclid's proof lies in the fact that the result is incredibly useful. Aside from their role in pure number theory, large prime numbers are used to make and break government codes.

"If the thing you're proving is useful or powerful and yet your proof is simple, that is a great thing," says Conway. A short, concise proof of a less important theorem might not be called elegant, he says. It would simply be cute or interesting.

But the most important quality of an elegant proof is that it makes you think, "Aha! How silly that I didn't think of that."

For example, consider a problem posed by mathematician Ron Graham. Consider a sequence of 101 distinct numbers arranged in any order you like. You can find a subsequence of 11 increasing or decreasing numbers in that set, he says.

First, to get a sense of it, think of the first 100 natural numbers arranged as follows:

91, 92, 93, …, 100, 81, 82, 83, …, 90,
71, 72, 73, …, 80, …, 1, 2, 3, …, 10.

This is a sequence of 100 numbers. We can pick one number from each "decade" to create a subsequence of 10 decreasing numbers, such as 95, 85, 75, 65, 55, 45, 35, 25, 15, 5. Or you can pick 10 increasing numbers. But it's impossible to find an increasing or decreasing subsequence of 11 numbers.

So intuitively you can believe that with 101 numbers there will be such a sequence of 11 numbers. But how can we prove it? The uninspired approach would be to check all cases. But we are going to use a cute—ahem, elegant—trick.

We assign to each of the 101 numbers $A_1, A_2, \ldots, A_k, \ldots, A_{101}$ a pair of integers (i_k, j_k) as follows. Let i_k be the length of the longest increasing subsequence ending in A_k. For example, if the sequence is <11, 3, 5, 1, 7, 2, …> and $k = 6$, then the largest increasing sequence ending in $A_k = 2$ is <1, 2> and $i_6 = 2$.

Similarly, let j_k be the length of the longest decreasing sequence ending in A_k. For our example, if $k = 6$, the longest decreasing sequence ending in $A_6 - 2$ could be either <11, 5, 2> or <11, 3, 2> or <11, 7, 2>. In any of these cases, $i_6 = 3$. So for $k = 6$, we have $A_k = (2, 3)$.

Now we can prove that no two pairs of these integers can be the same. For assume the contrary: suppose that $(i_m, j_m) = (i_n, j_n)$ for some subscripts m and n, with $n > m$. Now if $A_n > A_m$, then surely $i_n > i_m$, because otherwise you could just append A_n to the end of an increasing sequence measured by i_m. Similarly, if $A_n < A_m$, then $j_n > j_m$.

Now suppose all of the values for i_k and j_k are between 1 and 10. Then you would have 100 pairs. But you have 101 pairs, so the pigeonhole principle[1] guarantees that one of the pairs must contain an 11. This pair "points" to a subsequence such as is required in the problem.

Problem 2. Generalize this result to a sequence of $n^2 + 1$ different numbers.

Problem 3. Provide a counterexample to show that if the 101 numbers in the sequence are not distinct, then the result is false.

Don't worry if your proofs aren't elegant—most proofs aren't. But be sure to keep your eye peeled for an unexpected glimpse of God's book.

As John Conway put it: "Sometimes mathematics is like wandering around a strange town, wandering around some streets and suddenly you turn the corner and the view changes—you see the beauty of the whole thing." ◘

[1]See "Pigeons in Every Pigeonhole" in the January 1990 issue of *Quantum*.—*Ed.*

STICKING POINTS

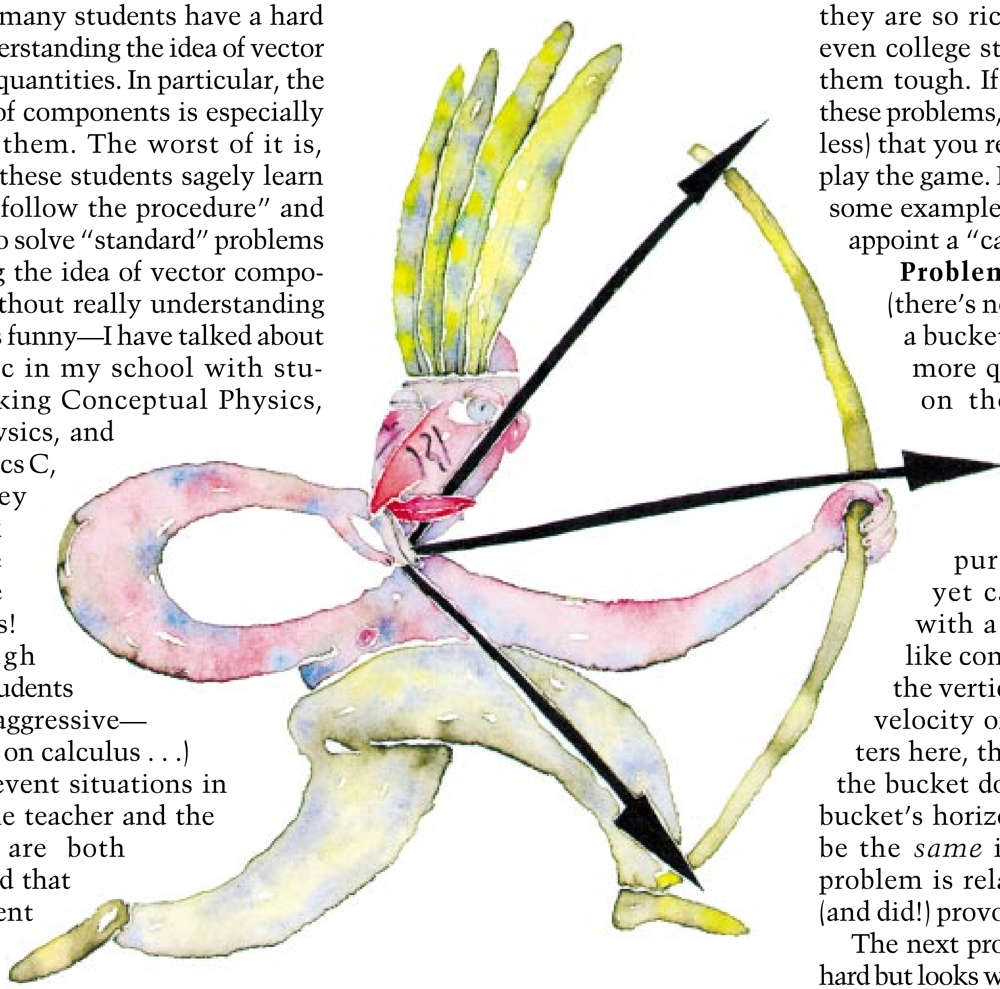
Important components of learning components

You use vectors—but do you really understand them?

by Boris Korsunsky

FROM MY TEACHING EXPERIENCE (both in Russia and the United States) I strongly believe that many students have a hard time understanding the idea of vector physical quantities. In particular, the concept of components is especially hard for them. The worst of it is, many of these students sagely learn how to "follow the procedure" and are able to solve "standard" problems involving the idea of vector components without really understanding them. It's funny—I have talked about this topic in my school with students taking Conceptual Physics, Intro Physics, and AP Physics C, and they all ask the same naïve questions! (Although the AP students are less aggressive—they rely on calculus . . .)

To prevent situations in which the teacher and the student are both convinced that the student actually *does* understand the idea while in fact he or she does *not*, I have used unusual, "tricky" problems that, as far as the math is concerned, are accessible even for an introductory high school physics course. At the same time they are so rich conceptually that even college students find many of them tough. If you're able to solve these problems, I can be sure (more or less) that you really do know how to play the game. In this article I'll offer some examples that are sure to disappoint a "calculus person."



Art by Sergey Ivanov

Problem 1. It's raining (there's no wind, though). Will a bucket be filled with water more quickly if it's resting on the ground or if it's placed on a horizontally moving platform?

The question is purely qualitative and yet can be solved easily with a "quantitative" tool like components. Since only the vertical component of the velocity of the raindrops matters here, the time needed to fill the bucket doesn't depend on the bucket's horizontal speed and will be the *same* in both cases. This problem is relatively easy but can (and did!) provoke a nice discussion.

The next problem is also not that hard but looks weird to many students.

Problem 2. A person is pulling a boat with a rope as shown in figure 1. At a certain moment the angle between the rope and the boat's velocity is θ. (You may say: "Well, first of all, that's impossible!" Is it?) The speed of the boat is v. Find the speed u at which the person must be pulling the rope at this moment.

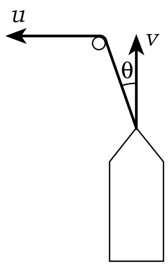


Figure 1

This problem also makes use of the idea of components. The answer is $u = v \cos \theta$. If you caught the drift of the problem, you would say that the component of the boat's velocity along the rope equals the velocity of the rope (we assume the rope doesn't stretch—otherwise the problem would be pointless). Unfortunately, from my experience many students are totally convinced that in order to be able to deal with components in a particular problem, you *must* have two *perpendicular* coordinate axes. This problem clearly says, "No, you don't."

The next problem looks different, but it's actually quite similar to problem 2.

Problem 3. A bar propped against a wall begins to slide down (fig. 2).

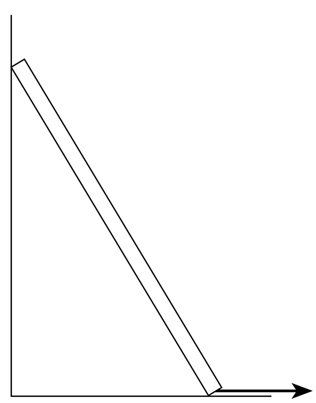


Figure 2

The velocity of the bottom end of the bar is given. Find the velocities of the top end of the bar and the middle of the bar graphically.

The way to solve this problem (which is admittedly a bit harder) is shown in figure 3. Since the bar is a rigid body, the components of the velocities of all points of the bar in the direction along the bar are the same. (Otherwise the distances between them would change.) If we know the component of a vector along a certain direction and the actual direction of the vector, we can easily find the vector itself. The direction of the velocity of the top end is obvious. How about the middle? Geometry tells us that as the bar slides down, the distance OC remains constant. This means that the midpoint C moves along an arc, and its velocity at all times is perpendicular to OC. Now we can "construct" the corresponding vector, as shown in figure 3.

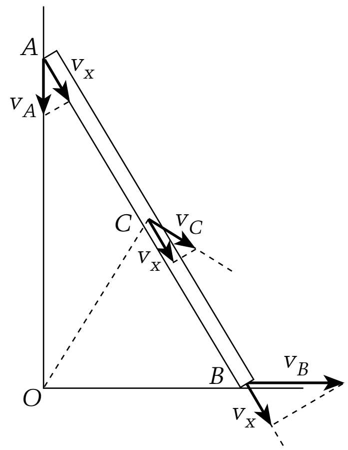


Figure 3

The next problem is really tricky and I'm sure you'll enjoy it. (That is, unless you're *too good* at math, which might cause problems!)

Problem 4. Four ninja turtles are ready for battle, standing at points A, B, C, D forming a square, as shown in figure 4. At the same moment they start to chase one another: the velocity of turtle A (sorry—I can never manage to remember their wonderful names!) is directed at all times toward turtle B, whose velocity, in turn, is directed toward turtle C, who is chasing turtle D in the same manner. And turtle D is chasing turtle A, of course. They all have the same speeds, and it's pretty obvious that, moving in curved lines, they eventually come together at the center of the initial square $ABCD$. How long will it take if the side of the initial square is L and the speed of each ninja turtle is v? (Bonus question: What's the point of such a contest?)

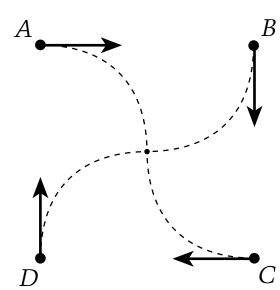


Figure 4

Isn't this a great problem? We certainly can't analyze these beyond-the-bounds-of-simple-math curves. Well, if you *can*, too bad—you'll miss all the fun! And the fun is to exploit the symmetry of the arrangement. At all times the turtles will form a square that decreases in size and simultaneously rotates. What a sophisticated motion! But the *center* of the square obviously does *not* move. And this is exactly where they meet—the point that interests us.

Now the components come into play. Although the direction of the velocity of each turtle changes continually, the component of the velocity of each turtle *directed toward the center* makes the same angle (45°) at all times with the velocity itself and, therefore, *retains its magnitude*, which is $v(\sqrt{2}/2)$. Now we get the answer right away. Isn't that great? (The answer is indeed L/v.)

Of course, components come in handy when we're faced with problems involving Newton's laws of motion. Here are a couple of nice examples.

Problem 5. The system shown in figure 5 is allowed to move freely from the state of rest with no friction. What will happen first: will block 1 hit the pulley, or will block 2 hit the wall?

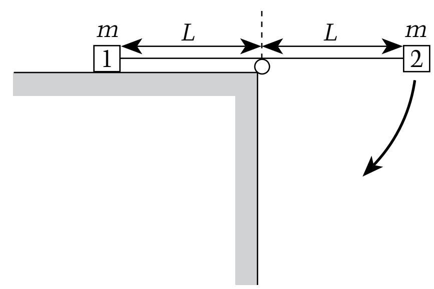


Figure 5

What can we do here? The direction and the magnitude of the force exerted on block 2 change continuously! Ready for some horrible integrating? Guess again!

This is one not-so-easy olympiad-style problem whose solution is amazingly short. Just consider the horizontal components. The force of tension of the string (which is certainly the same for both blocks) is the only one that contributes to the horizontal acceleration of both blocks. Of course, the horizontal component of this force is greater for block 1 *at all times*! Since both blocks have the same distance to go, block 1 will win the race. (The horizontal component of its velocity is *at all times* greater that that of block 2.)

The next (and last) problem brings in the idea of torques as well as components. (There's your hint!)

Problem 6. A uniform bar leans against a wall as shown in figure 6.

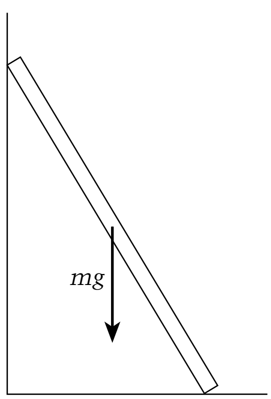


Figure 6

Given the fact that the wall is frictionless and given the vector representing the force of gravity acting on the bar, find the vector corresponding to the force of friction between the bar and the floor graphically. (Can the floor be frictionless, too?)

The solution is shown in figure 7. Two important ideas are involved. First, the net torque with respect to any point must be zero. Second, since the normal force of the wall and the force of gravity both "pass through" point A, the reactive force of the floor must also pass through the same point!

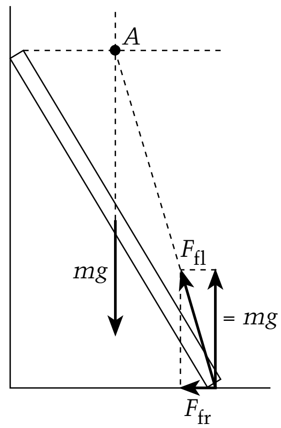


Figure 7

Now that we know the direction of this force, it's a good time to recall the fact that the vertical component of the floor's reactive force F_{fl} equals the force of gravity (which enables us to "construct" the vector corresponding to the floor's force). With this vector available to us, we can easily plot its horizontal component—which happens to be the unknown force of friction!

Tricky problems are a lot of fun and usually help us *really* understand a concept. I'll leave you with a few exercises. I'm sure you'll have a good time with them—eventually!

Exercises

1. A group of ants is pulling a small stick. At a certain moment the velocities of the ends A and B make the angles α and β, respectively, with the stick (fig. 8). The speed of end A is also given. Find the speed of end B.

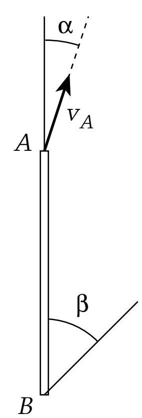


Figure 8

2. When the ants are done with the stick, they keep working hard. Now they are pulling a square piece of cardboard $ABCD$. At a certain moment it's known that the velocity of A equals v and is directed along AC. The velocity of C at this moment is directed along CD (fig. 9). Find the velocities of B, C, and D.

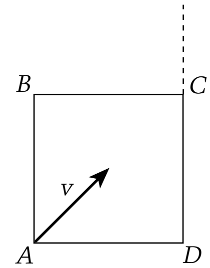


Figure 9

3. When the ants finish this bit of work, they take a break. (You're welcome to do the same!) After their siesta they pull a cardboard equilateral triangle ABC (fig. 10). It's known that at a certain moment the velocity of A is v and is directed along AB, whereas the velocity of C is directed along BC. Find the velocities of B and C.

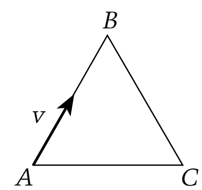


Figure 10

4. Why is it easier to pull a nail out of a board if you turn it continuously while pulling? (Hint: consider the component of the force of friction, which acts against the force you exert in pulling.)

Boris Korsunsky *teaches at Northfield Mount Hermon School in Northfield, Massachusetts.*

ANSWERS, HINTS & SOLUTIONS ON PAGE 59

The legacy of Norbert Wiener

Part II: Brownian motion and beyond

IN 1919 WIENER'S NOMADIC existence ended at last. He worked for a few months as a reporter for the *Boston Herald* and was fired. Finally, his father's friend Professor Osgood at Harvard interceded and obtained for Norbert an instructorship at MIT. In 1919, this was not a notable appointment. At the time, the mathematics department at MIT was purely a service department, valued only for its contribution to the engineering curriculum. Thus it is remarkable that MIT accommodated young Wiener, a man whose past experience did not recommend him as a teacher. In addition, even if MIT had sought prowess in mathematical research, Norbert Wiener in 1919 would not have been a strong candidate. He had published fifteen undistinguished articles on logic and nothing at all in traditional mathematics. But, whether MIT's decision to hire Wiener was guided by phenomenal insider information or was just a fortuitous product of the "old boy network," there can be no doubt that Wiener's appointment was a gamble that paid off for both parties. Wiener remained at MIT until his retirement in 1960, and during that period he not only put MIT on the map mathematically, he also played a profound part in the creation of the culture to which MIT owes much of its present fame and prestige.

Norbert Wiener in 1926.

At MIT the prodigy bloomed. Perhaps his emergence was an expression of his having at last found in mathematics his true calling; maybe it was the sense of security and self-esteem that came with a steady job; or possibly it was simply that, at age 24, the ex-prodigy had caught up with himself and was ready to become a genius. In any case, during his first dozen years at MIT, Wiener made his most astounding contributions to pure mathematics: he constructed Brownian motion, laid a new foundation for potential theory, and invented his generalized harmonic analysis.

The history of *Brownian motion* has taken some interesting twists and turns. The name honors the nineteenth-century botanist Robert Brown, who reported that pollen and many types of inorganic particles suspended in water perform a strange St. Vitus dance. Brown refuted some facile explanations of this motion, although debate still raged over whether the movement was of biological origin. It was Einstein's famous 1905 article on the subject that catapulted Brownian motion into twentieth-century physics. Einstein showed that a molecular (as opposed to a continuum) model of water predicts the existence of the phenomenon that Brown observed. Interestingly, he predicted Brownian motion before learning about Brown's observations.[1]

Part I appeared in the November/December 1994 issue. Reprinted from the program booklet for The Legacy of Norbert Wiener: A Centennial Symposium in Honor of the 100th Anniversary of Norbert Wiener's Birth, October 8–14, 1994, prepared by the MIT Department of Mathematics with the assistance of Tony Rothman.

[1] On page 17 of *Dynamical Theories of Brownian Motion* (Princeton University Press, 1967), Edward Nelson remarks, "It is sad to realize that despite all the hard work which had gone into the study of Brownian motion, Einstein was unaware of the existence of the phenomenon. He predicted it on theoretical grounds and formulated a correct quantitative theory of it." He quotes Einstein as saying, "My major aim . . . was to find facts which would guarantee as much as possible the existence of atoms of definite finite size."

Because it is virtually impossible to solve Newton's equations of motion for anything like the number of particles in a drop of water, Einstein adopted a statistical approach and showed that the evolution of the distribution of Brownian particles is governed by the heat equation. That is, the density of particles at each point follows the same physical law as the temperature at each point. Actually, from the physical point of view, this description of Einstein's paper throws out the baby with the wash. A physicist cannot talk about a one-size-fits-all *heat equation* any more than a one-size-fits-all *wave equation*; there are all-important constants that enter any physical equation. For the wave equation, the essential physical constant is the speed of light. In the case of the heat equation, there is the *diffusion constant*, and it was Einstein's formula for the diffusion constant that won his 1905 article its place in history. Namely, Einstein expressed the diffusion constant as the ratio of several physical quantities, one of which was Avogadro's number.[2] It turns out that, with the exception of Avogadro's number, all these quantities, including the diffusion constant itself, were either known or measurable experimentally. Thus, his formula led to the first accurate determination of Avogadro's number.

If one ignores physics and analyzes Einstein's model from a purely mathematical standpoint, what Einstein was saying is summarized by the following three assertions about the way in which Brownian particles move.

1. Brownian particles travel in such a way that the behavior over two different time intervals is independent. Thus, there is no way to predict future behavior from past behavior.

2. The particle is equally likely to move in any direction, and the distance traversed by a Brownian particle during a time interval is on average proportional to the square root of the time.

3. The trajectories of Brownian particles are continuous.

With reasonably standard results from the modern theory of probability, one can deduce from Einstein's three assumptions the conclusion that the distribution of Brownian particles evolves according to a heat equation. (The all-important diffusion constant is determined by the proportionality constant in assertion 2.) Of course, in 1905, a mathematically satisfactory formulation of probability theory had yet to be given. Thus, Einstein's derivation was, mathematically speaking, rather primitive. Moreover, implicit in his model was an important mathematical challenge: *the verification that one can construct a distribution on the space of trajectories so that assertions 1, 2, and 3 are satisfied.*[3]

At the turn of the century, the French school of analysis was hard at work creating the subject that we now call *measure theory* (that is, the theory by which we assign volume to sets).[4] The French school, especially E. Borel and H. Lebesgue, freed measure theory from its classical origins and made it possible to consider the problem of assigning *probabilities* to subsets of trajectories. However, in spite of their many magnificent achievements, neither Borel, Lebesgue, nor their disciples like P. Lévy, S. Banach, M. Fréchet, and A. N. Kolmogorov had been able to mathematically rationalize Einstein's model of Brownian motion. All of them were well aware of the essential problem, but none of them had been able to carry out the required construction. This was the problem that Wiener solved.

In hindsight, Wiener's strategy looks a little naïve. In particular, he completely circumvented the issues on which more experienced mathematicians had foundered. In a marvelous demonstration of the power of optimism, he supposed that the desired assignment of probabilities could be made and asked how this assignment would look in a cleverly chosen coordinate system. He then turned the problem around and showed that the coordinate description leads to the existence of the desired assignment. (This general line of reasoning is familiar to anyone who has ever solved a problem by saying "let x be the solution" and then found x as a consequence of the properties that it must have.) Wiener's Gordian-knot solution to the problem enhances its appeal, and the assignment of probabilities at which Wiener arrived in "Differential Space" has, ever since, borne his name. It is called *Wiener measure.*

The importance of Wiener measure is hard to exaggerate. It represents what we now dutifully call a *paradigm*. For one thing, its very

[2]Avogadro's number is a universal constant measuring the number of molecules in a gas per unit volume at a fixed pressure. It can also be defined as the number of atoms in one gram of hydrogen.

[3]Actually, Einstein's 1905 article was not the first one in which this problem appears. Five years earlier, H. Poincaré's brilliant student L. Bachelier came to the conclusion that the fluctuation of prices on the Paris Bourse follow trajectories whose distribution satisfies assertions 1, 2, and 3. It was not until the 1970s that the economics literature on this subject converged with the engineering and mathematical literature. The result is a much more sophisticated way to calculate risk in large financial markets, which has become an indispensable tool for loan, investment, and trading companies. Finally, one should remark that Bachelier, as distinguished from Einstein, really addressed the problem of computing the probability of nontrivial events that can be formulated only in the path–space context. The first physicist to address such problems was M. Smoluchowski, who used an approximation scheme based on random walks.

[4]Prior to their efforts, the only available theory was basically the one introduced by Archimedes, rediscovered by Fermat and Newton, and now forced on every calculus student. Of course, that theory had been tightened up by Cauchy, Riemann, and others, but it was still seriously deficient. For example, one could not show that the whole is the sum of its parts unless there were at most finitely many parts. In addition, although Riemann's theory served quite well in finite dimensional contexts, there was no theory at all for infinite dimensional spaces, like the space of all Brownian trajectories.

existence opened a floodgate and led Lévy, Kolmogorov, and others to create the theory of *stochastic processes*, thereby ushering in the modern theory of probability. In addition, Wiener measure is, in a sense that can be made very precise, as universal as the standard Gaussian (or normal) distribution on the real line: it is the distribution that arises whenever one carries out a central limit scaling procedure on path–space valued random variables.[5] This is the underlying reason why Wiener measure arises as soon as one is studying a phenomenon that displays the properties 1, 2, and 3. It is also the reason why, again and again, Wiener measure comes up in models of situations in which one is observing the net effect of a huge number of tiny contributions from mutually independent sources—as in the motion of a pollen particle, the Dow Jones average, or, as Wiener himself observed, the distortions in a signal transmitted over a noisy line.

Although his construction of Brownian motion was Wiener's premiere achievement during the period, it was not his only one. In a sequence of articles from 1923 through 1925, Wiener also looked at a fundamental problem in the theory of electrostatics. The problem was to decide what shape electrical conductor can carry a fixed charge. Zaremba had shown that certain conductors in the shape of spikes are unable to carry charge—they discharge spontaneously at the tip. (The reverse of this phenomenon is what makes a lightning rod work.) On the other hand, Zaremba had shown that cone-shaped conductors do hold their charge. In the mathematical model spontaneous discharge corresponds to an abrupt change—a *discontinuity*—in the voltage across the interface between the conductor and the surrounding medium. The electrical field has a constant voltage on the conductor, and the equilibrium is stable (no sparks) if the voltage is *continuous* across the interface.

Wiener described all shapes for which instability occurs and established a new framework for the entire subject of potential theory. In sharp contrast with many models in mathematical physics, he showed that the voltage in equilibrium is well defined mathematically, regardless of whether the conductor is stable or not. He then formulated a wholly original test, now known as the *Wiener criterion*, that determines at which points the voltage is discontinuous. A key step in Wiener's approach was to extend to arbitrary shapes a classical notion known as electrostatic capacity.[6] He used a procedure that is analogous to, but more intricate than, the one invented by Lebesgue when he assigned a volume to regions for which there was no classical notion of volume. Indeed, Wiener's capacity is closely related to, but more subtle than, the measures used for fractals.[7]

Another topic that Wiener investigated during this period was what we now call distribution theory or the theory of generalized functions. Not long after Wiener arrived at MIT, Professor Jackson and other members of the electrical engineering department at MIT asked Wiener to develop a proper foundation for the Heaviside calculus—a calculus for solving differential equations by means of Fourier and Laplace transforms. Heaviside's calculus transforms a differential equation into an equation involving multiplication, as in $Ax = B$. To solve for x, we simply divide: $x = B/A$. The difficulty is that this easy formula for the solution then has to be transformed back into a meaningful statement about the solution to the original differential equation. This involves making sense of the inverse of the Fourier–Laplace transform. Wiener undertook the description of how multiplication and division correspond to the operations of differentiation and integration. Laurent Schwartz, the father of the theory of distributions, acknowledges that Wiener's treatment in 1926 anticipated all others by many years.

Just as the physics of Brownian motion had stimulated Wiener to profound new mathematics, so the practical problem of processing electrical signals led him to a deep extension of classical Fourier analysis. Fourier analysis consists of decomposing a periodic signal into a sum of pure sine waves. The fundamental formula of Fourier analysis—the Parseval formula—says that the total energy of the signal in each period is the sum of the energies of its pure waves. The collection of frequencies at which these amplitudes occur is known as the spectrum of the signal, and these come from a discrete list of values—the harmonics of a vibrating string. There is a similar fundamental formula due to Plancherel for the decomposition of nonperiodic waves that measures the total energy over all time. The spectrum of the signal is spread over the continuum of frequencies, and the formula measures the amount of energy of the signal concentrated in a given band of frequencies. The problem is that the signals that occur in practice in electrical systems do not fit into the frame of either of these theories. The signals are not periodic and the spectrum is not confined to a special list, so that Fourier series are inadequate. On the other hand, the total energy over an infinite time period is infinite, so that Plancherel's theory does not apply. Wiener overcame this difficulty with what he named generalized harmonic analysis. Wiener took as his starting place certain autocorrelation numbers, which compare the signal to the same signal with a time delay. These

[5]A full understanding of this universality came only in the 1950s and was provided by P. Lévy, R. H. Cameron, M. Donsker, P. Erdős, M. Kac, W. T. Martin, and I. E. Segal.

[6]The electrostatic capacity of a conductor can be defined as the total charge carried by the conductor in equilibrium when the voltage difference between the conductor and its surroundings is fixed at, say, one hundred volts.

[7]There is an amusing irony associated with Wiener's investigations into potential theory. Namely, as S. Kakutani discovered in the early 1940s, potential theory is related to Brownian motion in deep and wonderful ways. Wiener completely missed this beautiful and useful connection with his previous work.

were precisely what could be measured in practice. Then, instead of dealing with total energy, Wiener considered the average energy of the signal over a long time interval. His theory was flexible enough to encompass both periodic signals and signals composed of a continuum of frequencies, such as "white noise."

One of the key ingredients in Wiener's generalized harmonic analysis was a new method to calculate limits of averages. His first step was to rephrase the problem so that it became one of determining when two different weighted averages are very close to each other. The recast problem fit into the general framework of so-called Tauberian theory—a theory to which Hardy and Littlewood had made several contributions. But instead of using some refinement of the techniques of his teachers, Wiener introduced a new approach that not only solved his own problem but revealed the fundamental mechanism of all previous problems of this type.[8] In his monograph on the subject, Wiener illustrates his ideas with an elegant proof of the Prime Number Theorem, one of the most beautiful applications of analysis to number theory.

With the publication of his work on generalized harmonic analysis and Tauberian theorems, Wiener's reputation was at last established. In 1932 he was promoted to Full Professor at MIT with a salary of $6,000. The following year, he was elected to the National Academy of Sciences, and he won the Bôcher Prize, a prize given every five years for the best work in analysis in the United States.

The major works outlined above by no means exhaust Wiener's intellectual activity. Throughout the 1930s he continued to expand on harmonic analysis, with the same engineering applications clearly in view. He wrote an influential book with R. E. A. C. Paley and a seminal paper on integral equations with E. Hopf. He made excursions into quantum mechanics with Max Born and sorties into five-dimensional relativity (Kaluza–Klein theory) with Dirk Struik. In the late 1930s Wiener made a significant contribution to the mathematical foundations of statistical mechanics by extending G. D. Birkhoff's 1931 ergodic theorem. His 1938 paper "The Homogeneous Chaos," which undertakes to fathom nonlinear random phenomena, has descendants in constructive quantum field theory, under the name "Wick ordering."

[8] Wiener's work led to I. M. Gelfand's far-reaching formulation of a notion of spectrum that can be used to analyze multiplication and division in any algebraic system.

The concluding segment of this centenary essay will cover Wiener's work on the control of anti-aircraft fire during World War II and his most famous legacy—cybernetics.

TO BE CONTINUED IN THE NEXT ISSUE

WINNER OF THE EXCELLENCE IN JOURNALS DESIGN & PRODUCTION AWARD (1993) OF THE ASSOCIATION OF AMERICAN PUBLISHERS AND A 1995 FOLIO *EDITORIAL EXCELLENCE AWARD*

QUANTUM

We hope this *Quantum* sampler has given you some idea of what this lively, handsomely illustrated bimonthly magazine is all about. In addition to its feature articles, *Quantum*'s departments include At the Blackboard (the beauty and usefulness of equations), In the Lab (hands-on science), Toy Store (mathematical amusements), Kaleidoscope (a collection of snippets designed to consolidate your grasp of a given topic), How Do You Figure? (challenging problems in physics and math), Brainteasers (fun problems requiring a minimum of math background), Looking Back (biographical and historical pieces), and Gallery Q (an exploration of links between art and science). But even this list isn't exhaustive.

As you have seen, *Quantum* actively engages you, posing questions and pursuing ideas as if they're brand new (even if they're a thousand years old). Most *Quantum* articles include problems for you to work through, and each issue contains an answer section. Some articles are elegant expositions of sophisticated concepts, and some give an unexpected twist to a well-known idea or phenomenon. Others show that there is no such thing as a silly question. For instance, one article asks: "Why are the holes in Swiss cheese round?" *Quantum* in effect says: smirk at your own risk—there's *knowledge* to be had in such "naive" questions!

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Quantum is published by the National Science Teachers Association (NSTA) and Springer-Verlag New York, Inc. NSTA is responsible for the editorial side of the magazine, and Springer prints and distributes it. NSTA works closely with Quantum Bureau of the Russian Academy of Sciences in selecting and preparing material, and *Quantum*'s advisory board consists of members of the American Association of Physics Teachers (AAPT) and the National Council of Teachers of Mathematics (NCTM).

Where did *Quantum* come from?

Quantum came from Russia! It's based on a Russian magazine of math and physics called *Kvant*. We use translations from *Kvant* as well as original material generated in the US. The first issue of *Quantum* appeared in January 1990.

What is *Kvant*?

Kvant is a journal of math and physics founded in 1970 by two prominent Soviet scientists: the mathematician A. N. Kolmogorov and the physicist I. K. Kikoyin. (*Kvant* means "quantum" in Russian.) It is an outgrowth of a Russian educational tradition that encourages top-notch scientists to extend their teaching to high school students and to write challenging but accessible material for them. *Kvant* helped form an entire generation of Soviet scientists, many of whom now write for *Kvant*/*Quantum*.

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